Kamnitzer

$\mathfrak{g}$: simple Lie algebra $/\mathbb{C}$

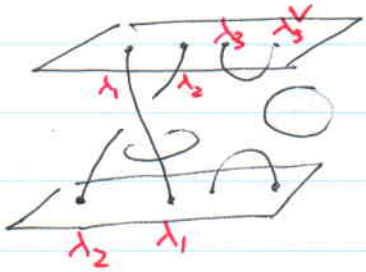
Λ_+ : dominant weight $\ni \lambda$ $\quad V_\lambda$: irr. of $\mathfrak{g}$

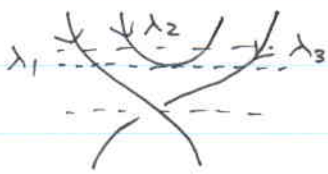
RT invariant :

$$\left\{ \begin{array}{l} \text{labbelled tanges} \\ \text{endpoints } (\lambda_1, \dots, \lambda_n) \\ \qquad (\mu_1, \dots, \mu_m) \end{array} \right\} \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, V_{\mu_1} \otimes \cdots \otimes V_{\mu_m})$$

$$T \longmapsto \psi(T)$$

tangles : embedding $S^1 \sqcup S^1 \sqcup \dots \sqcup S^1 \sqcup I \sqcup \dots \sqcup I \longrightarrow \mathbb{R}^2 \times [0,1]$





$(\lambda_1, \lambda_2, \lambda_2^\vee, \lambda_3)$

(λ_3, λ_1)

$$\psi(P) : V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_2^\vee} \otimes V_{\lambda_3} \to V_{\lambda_1} \otimes V_{\lambda_3} \to V_{\lambda_3} \otimes V_{\lambda_1}$$

$$v \otimes w \mapsto w \otimes v$$

used a projection of T to define $\psi(T)$, but actually it is a tangle invariant.

$\rightsquigarrow$ boring invariant

change $\mathfrak{g} \rightsquigarrow U_q\mathfrak{g}$ and use R-matrix

If L link ($=$ tangle with no end points)

$$\mathbb{C}[q, q^{-1}] \xrightarrow{\psi(L)} \mathbb{C}[q, q^{-1}]$$

$\psi(L)(1)$ is a link invariant

$\mathfrak{g} = \mathfrak{sl}_2$, rep. = std rep. $\Rightarrow$ Jones polynomial

<u>Idea</u> (Khovanov) Crane-Frenkel

categorify this setup

(1) replace each $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ with a graded triangulated category $D(\lambda_1, \cdots, \lambda_n) \circlearrowleft \{1\}$

(2) replace $\Psi(T)$ with a $\overbrace{\text{functor}}^{\text{exact}}$

$$\Phi(T) : D(\lambda_1, \cdots, \lambda_n) \to D(\mu_1, \cdots, \mu_m)$$

s.t. $K_{\mathbb{C}}(D(\lambda_1, \cdots, \lambda_n)) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$

and

$$\begin{array}{ccc} K_{\mathbb{C}}(D(\lambda_1, \cdots, \lambda_n)) & \cong & V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \\ {[\Phi(T)]} \downarrow & \circlearrowright & \downarrow \Psi(T) \\ K_{\mathbb{C}}(D(\mu_1, \cdots, \mu_m)) & \cong & V_{\mu_1} \otimes \cdots \otimes V_{\mu_m} \end{array}$$

$\mathbb{C}[q, q^{-1}]$-module $\qquad q$ acts from the shift functor.

<u>Question</u>

① Does such a categorification exist?

② Is there a natural way to construct such categorifications?

First assume that all λ_i are minuscule representations

e.g. $\Lambda^k \mathbb{C}^n$ $\forall k, n$

minuscule rep of $\mathfrak{sl}_n$

$\exists$ smooth proj var. $\quad Gr^{\lambda_1} \tilde{\times} \cdots \tilde{\times} Gr^{\lambda_n} \curvearrowleft \mathbb{C}^*$

(iterative prod. of Grassmann bdle)

s.t. $H_*(Gr^{\lambda_1}\tilde{\times}\cdots\tilde{\times}Gr^{\lambda_n}) \cong V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}$

(geometric Satake)

$D^b(Coh^{\mathbb{C}^*}(Gr^{\lambda_1}\tilde{\times}\cdots\tilde{\times}Gr^{\lambda_n})) =: D(\lambda_1,\ldots,\lambda_n)$

objects : complexes of coherent sheaves

morphisms : homotopy classes of complexes

localize at q. i

$K(D(Gr^{\lambda_1}\tilde{\times}\cdots\tilde{\times}Gr^{\lambda_n})) \cong H_*(Gr^{\lambda_1}\tilde{\times}\cdots\tilde{\times}Gr^{\lambda_n}) \cong V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}$

$\Phi(T)$ functors

roughly describe them in general.

$\Phi(\frown)$, $\Phi(\times)$, $\Phi(\smile)$

do not know to check relation in general.

Now pass to sl_2, $V_\lambda = \mathbb{C}^2$

$\underbrace{Gr^{\lambda_1}\tilde{\times}\cdots\tilde{\times}Gr^{\lambda_1}}_{n} =: Y_n$

NB. non minuscule
--→ need to categorify IC

N : large $(\mathbb{C}^2)^N$ be v.s. with basis $e_1,\ldots,e_N$
$f_1,\ldots,f_N$

$z : (\mathbb{C}^2)^N \circlearrowleft$ $\quad ze_i = e_{i-1}$ $\quad ze_1 = 0$

(N,N)-nilpotent $\quad zf_i = f_{i-1}$ $\quad zf_1 = 0$

$$Y_n = \{ 0 = L_0 \subset L_1 \subset \cdots \subset L_n \subset \mathbb{C}^{2n} \mid \dim L_i = i,\ zL_i \subset L_{i-1} \}$$

Springer fiber of a partial flag

$\underline{\text{Ex.}}$ $Y_1 = \mathbb{P}^1$

$\quad \Vert$

(1-dim sub sp. in $\operatorname{Ker} z$)

$Y_n \to Y_{n-1}$ forgetting the last piece $\quad \mathbb{P}^1$-b'dle

$$L_{n-1} \subset L_n \subset z^{-1}L_{n-1}$$

differ by 2

$\dim Y_n = n$

$Y_2 = F_2$: Hirzebruch surface

$\mathbb{C}^* \curvearrowright \mathbb{C}^{2N}$ $\quad$ $t e_i = t^i e_i$, $\quad t f_i = t^i f_i$ $\quad \curvearrowright Y_n$

$$X_n^i = \{ L_\bullet \in Y_n \mid z L_{i+1} = L_{i-1} \} \overset{j}{\subset} Y_n \quad \text{codim } 1$$

$$\begin{array}{ccc} L_{i+1} & & \\ \downarrow z & & \\ L_i & \Big) z & \\ \downarrow z & & \\ L_{i-1} & & \end{array}$$

$X_n^i \xrightarrow{q} Y_{n-2}$ $\quad \mathbb{P}^1$-b'dle

$(L_1, \cdots, L_n) \mapsto (L_1, \cdots L_{i-1} = zL_{i+1}, zL_{i+2}, \cdots, zL_n)$

$\mathcal{E}_i$: line b'dle on Y_n whose fiber at $(L_1, \cdots, L_n)$ is L_i / L_{i-1}

$\Vert \cap \Vert$ $\quad G_n^i : D(Y_{n-2}) \to D(Y_n)$

$$\mathcal{F} \mapsto j_*(q^*\mathcal{F} \otimes \mathcal{E}_i)$$

$||\cup||$ $\quad F_n^i : D(Y_n) \longrightarrow D(Y_{n-2})$

$$\mathcal{F} \longmapsto i_*(j^*\mathcal{F} \otimes \mathcal{E}_{i+1}^\vee)$$

$$Z_n^i = \{ (L_\bullet, L'_\bullet) \in Y_n \times Y_n \mid L_j = L_j' \text{ for all } j \neq i \} \subset Y_n \times Y_n$$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & L_{i+1}=L'_{i+1} & \\
L_i \arrow[ur, dash] & & L'_i \arrow[ul, dash] \\
 & L_{i-1}=L'_{i-1} \arrow[ul, dash] \arrow[ur, dash] \arrow[d, dash] & \\
 & L_{i-2}=L'_{i-2} &
\end{tikzcd}$$

$$T_n^i(1) : D(Y_n) \to D(Y_n)$$

$||\times||$ $\quad \mathcal{F} \longmapsto \pi_{2*}(\pi_1^*\mathcal{F} \otimes \mathcal{O}_{Z_n^i})$

$$T_n^i(2) : D(Y_n) \to D(Y_n)$$

$$\mathcal{F} \longmapsto \pi_{2*}\left(\pi_1^*\mathcal{F} \otimes \mathcal{O}_{Z_n^i} \otimes \pi_1^*(\mathcal{E}_{i+1}^\vee) \otimes \pi_2^*(\mathcal{E}_i)\right)$$

$||\times||$ $\quad$ (adjoint)

$$\begin{tikzcd}[column sep=small]
 & Y_n \times Y_n \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y_n & & Y_n
\end{tikzcd}$$

$-\!\!\times\!\!- = ||$ $\quad$ equivalence

mutually inverse

Th
① $\Phi(\overset{i\ \ i+1}{||\times||}\cdots|) = T_n^i(1)$

$\Phi(||\times||\cdots|) = T_n^i(2)$

$\Phi(||\cup|||) = F_n^i$

$\Phi(|||\cap||) = G_n^i$

extends to a map

$\Phi : \{(n,m) \text{ tangles}\} \to \left\{ \begin{array}{l} \text{iso. class} \\ \text{of functors} \\ D(Y_n) \to D(Y_m) \end{array} \right\}$

$= |$ $\qquad F_n^i \circ G_n^{i+1} = \mathrm{id}$ $\qquad)(= ||$

$= |$ $\qquad F_n^{i+1} \circ T_n^i \circ G_n^{i+1} = \mathrm{id}$ $\qquad T_n^i(2) \circ T_n^i(1) = \mathrm{id}$

R3

② $$\begin{tikzcd} K(D(Y_n)) \arrow[r, phantom, "\cong"] \arrow[d, "\Phi(T)"'] & V^{\otimes n} \arrow[d, "\Phi(T)"] \\ K(D(Y_n)) \arrow[r, phantom, "\cong"] & V^{\otimes n} \end{tikzcd}$$ $\qquad V = \mathbb{C}[q, q^{-1}]^{\oplus 2}$

③ L is a link

$$\Phi(L) : D(Y_0) \circlearrowleft$$
$$D(Y_0) = D(\text{gr vector space})$$

$$H^{i,j}((\Phi(L))(\mathbb{C})) = H^{i+j,j}_{Kh}(L).$$

relation to Seidel-Smith

Y_{2n}

$$F_n = \{(L_1, \cdots, L_{2n}) : L_{2n} = \mathbb{C}^{2n}\} \subset \underbrace{\{L_\cdot \in Y_{2n} : pr : L_{2n} \to \mathbb{C}^{2n} \text{ is an isom.}\}}_{= U_n} \subset Y_{2n}$$

$$\mathbb{C}^{2n} = \mathrm{Span}\begin{pmatrix} e_1 \cdots e_n \\ f_1 \cdots f_n \end{pmatrix}$$

$$L_1 \subset L_2 \subset \cdots \subset L_{2n} \subset \mathbb{C}^{2N} = \mathrm{Span}\begin{pmatrix} e_1 \cdots e_N \\ f_1 \cdots f_N \end{pmatrix}$$

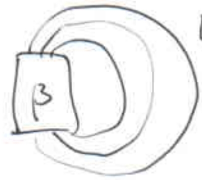
F_n = Springer fiber to the $(n.n)$-nilpotent

$n=1$

$$\mathbb{P}^1 \subset T^*\mathbb{P}^1 \subset Y_2$$
$$F_1 = \mathbb{P}^1, \quad U_1 = T^*\mathbb{P}^1, \quad Y_2 = \text{Hirz. surface}$$

L is a link and we choose a proj. of L $\quad \beta \in B_n$

$$H^{i,j}(\Phi(L)(\mathbb{C})) = \mathrm{Ext}^{i,j}_{Y_{2n}}(A, \beta(A))$$



L

A : the structure sheaf of a component of F_n
(doesn't depend on L)

$\underline{\text{RHS}} = \mathrm{Ext}^{i,j}_{U_n}(A, \beta(A))$

In Seidel-Smith

have a symplectic mfd M_n

and a braid group action $B_{2n} \curvearrowright \mathrm{Fuk}(M_n)$

Their knot invariant $HF^{\cdot}(A', \beta(A'))$

$A \subset M_n$ lagrangian

U_n is hyperkähler and is the same as M_n

~~HHS~~ more easy version motivated by physics

$HF^{\cdot}(A', \beta(A')) \overset{\text{conj}}{=} \mathrm{Ext}^{\cdot}(A, \beta(A))$

forgetting $\mathbb{C}^*$-action